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# Reaction, Trapping, and Multifractality in One-Dimensional Systems

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In the first part of this paper, we present two variants of the $A + A \to A$ and $A + A \to P$ reaction in one dimension that can be investigated analytically. In the first model, pairs of neighboring particles disappear reactively at a rate which is independent of their relative distance. It is shown that the probability density $\varphi(x)$ for a nearest neighbor distance equal to $x$ approaches the scaling form $\varphi(x) \sim c \exp(-cx/2)/(cx)^{1/2}$ in the long-time limit, with $c$ being the concentration of particles. The second model is a ballistic analogue of the coagulation reaction $A + A \to A$. The model is solved by reducing it to a first-passage-time problem. The anomalous relaxation dynamics can be linked in a direct way to the fractal time properties of random walks. In the second part of this paper, we discuss the complications that arise in systems with disorder. We present a new approach that relates first-passage-time characteristics in a one-dimensional random walk to properties of random maps. In particular, we show that Sinai disorder is a borderline case for the appearance of multifractal properties. Finally, we apply a previously introduced renormalization technique to calculate the survival probability of particles moving on the line in the presence of a background of imperfect traps.

**KEY WORDS:** Trapping, multifractality; one-dimensional systems.

## 1. INTRODUCTION

The classical rate laws for chemical kinetics can only be expected to apply in the case that the reactions do not strongly disturb the state of local equilibrium. Recently, the deviations from classical rate laws have been studied in detail for diffusion-limited binary reaction schemes[1-25] such as

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$A+T \to T$ (trapping problem), $A+A \to A$ (coagulation problem), and $A+A \to P$ and $A+B \to P$ (annihilation problems). In these models, the stationary state is a highly nonequilibrium state, corresponding to the absence altogether of any particles. As this state is approached in the long-time limit, nonequilibrium correlations build up, and possibly extend over an infinite range. As a result, the probability for a chemical encounter between two particles depends on the microscopic structure, i.e., on the specific form of the long-range correlations. In fact, the very existence of rate laws, with only the rate constants being determined by the microscopic structure, is no longer guaranteed. The existence of long-range correlations was also reported some time ago for the case of chemical reaction with a high activation barrier, although the range of these correlations remains finite and classical rate laws still apply.[26]

In the first part of this paper (Sections 2 and 3), we present two variants of the $A+A \to A$ and $A+A \to P$ reaction in one dimension that can be investigated analytically. In the first model, pairs of neighboring particles disappear reactively at a rate which is independent of their relative distance. It is shown that the probability density $\varphi(x)$ for a nearest neighbor distance equal to $x$ approaches the scaling form $\varphi(x) \sim c\exp(-cx/2)/(cx)^{1/2}$ in the long-time limit, with $c$ being the concentration of particles. The second model is a ballistic analogue of the coagulation reaction $A+A \to A$. The model is solved by reducing it to a first-passage-time problem. The anomalous relaxation dynamics can be linked in a direct way to the fractal time properties of random walks.

In Section 4, we show how a previously introduced renormalization technique can be used to calculate the survival probability of particles moving on the line in the presence of perfect traps and a background of imperfect traps.

Finally, in Section 5, we discuss in detail the first-passage-time characteristics in a one-dimensional random walk with disorder. It has been suggested that multifractal properties have to be invoked to understand, for example, the growth probability in diffusion-limited aggregation. We present a new and elegant approach that allows to discuss in detail the origin of multifractality in the much simpler situation of a one-dimensional random walk with (quenched) disorder. In this case, the probability for a walker to reach its next nearest neighbor is a random variable. We show that Sinai disorder is the borderline case separating the types of disorder for which first passage is a certain event and those for which the first passage probability is no longer equal to 1 (with probability 1), but displays multifractal properties.

## 2. INTERCHANGE REACTIONS IN ONE DIMENSION

### 2.1. Single-Particle Annihilation

It is well known that a random distribution of points on a line is characterized by an exponential probability density $\varphi(x)$ for the distance $x$ between neighboring particles,[27]

$$\varphi(x) = ce^{-cx} \tag{2.1}$$

where $c$ represents the concentration of the particles, which is also equal to the inverse of the average distance between the particles, $c = \langle x \rangle^{-1}$. Moreover, the distances between different pairs of particles are independent random variables. One way to generate such a random distribution is to add particles, one by one, at random locations on the line. The question we will address here is what happens if we do the reverse, i.e., if we eliminate randomly chosen particles. This elimination can be viewed as the result of a "chemical reaction" or "decay" of the particles under consideration. Since the elimination of a certain number of particles can be viewed as the previously mentioned process of adding particles at random locations, but terminated at an earlier time, we expect that an exponential density will prevail. We now proceed to show under which conditions this is indeed the case.

We consider a distribution of points on the line with a given probability density $\varphi(x, t=0)$ for the distance $x$ between neighboring particles at time $t=0$. Moreover, we assume that the distances between different neighboring pairs of particles are independent random variables at $t=0$. We now let single particles disappear at a constant rate $R$, and study the evolution of the interparticle distance density $\varphi(x, t)$. A first observation is that the distances between different pairs of particles remain independent random variables, characterized by a common probability density $\varphi(x, t)$ at time $t$. The equation of evolution for $\varphi(x, t)$ can be derived as follows. At time $t+dt$, we pick at random a particle, and calculate the probability $\varphi(x, t+dt)$ that its right nearest neighbor lies at a distance $x$. This can be realized, to lowest order in $dt$, in two different ways. Either the nearest neighbor at time $t$ was at the distance $x$, and was not removed in the time interval $dt$, or the nearest neighbor at time $t$ was removed in the time interval of length $dt$, but the next nearest neighbor was at a (total) distance $x$. We thus obtain

$$\varphi(x, t+dt) = R\,dt \int_0^x \varphi(x', t)\,\varphi(x-x', t)\,dx' + \varphi(x, t)(1 - R\,dt) \tag{2.2}$$

or

$$\partial_t \varphi(x, t) = R\left[\int_0^x \varphi(x', t)\, \varphi(x - x', t)\, dx' - \varphi(x, t)\right] \tag{2.3}$$

By introducing the spatial Laplace transform

$$\tilde{\varphi}(s, t) = \int_0^\infty e^{-sx} \varphi(x, t)\, dx \tag{2.4}$$

one can easily solve this equation. The solution reads

$$1 - \frac{1}{\tilde{\varphi}(s, t)} = e^{Rt}\left[1 - \frac{1}{\tilde{\varphi}(s, t=0)}\right] \tag{2.5}$$

Based on this exact result, the following conclusions can be drawn:

(1) If the initial density is exponential,

$$\varphi(x, t=0) = c(t=0)\, e^{-c(t=0)x} \tag{2.6}$$

or

$$\tilde{\varphi}(s, t=0) = \frac{1}{1 + s/c(t=0)} \tag{2.7}$$

then it remains so for all times:

$$\varphi(x, t) = c(t)\, e^{-c(t)x} \tag{2.8}$$

with rate equation

$$\partial_t c = -Rc \tag{2.9}$$

(2) If the initial density has a finite first moment

$$\langle x(t=0)\rangle = c^{-1}(t=0) = \int_0^\infty x\varphi(x, t=0)\, dx < \infty \tag{2.10}$$

then

$$\tilde{\varphi}(s, t=0) = 1 - s/c(t=0) + \cdots \tag{2.11}$$

Hence, the density approaches an exponential density in the long-time limit. More precisely, one has that [cf. (2.5) and (2.10)]

$$\lim_{t\to\infty} \tilde{\varphi}(sc(t), t) = \frac{1}{1+s} \tag{2.12}$$

with

$$c(t) = e^{-Rt} c(t=0) \tag{2.13}$$

In real space, one has the following scaling result:

$$\varphi(x, t) \overset{t \to \infty}{\approx} c(t)\, e^{-c(t)x} \tag{2.14}$$

In this sense, the exponential density is the attractor of all densities with finite first moment under the procedure of removing randomly chosen single particles.

(3) If the initial density is characterized by a fractal dimension $0 < \alpha < 1$, i.e.,

$$\tilde{\varphi}(s, t=0) = 1 - s^{\alpha} \tag{2.15}$$

(where we have chosen the coefficient of $s^{\alpha}$ equal to 1, for simplicity) it follows that

$$\lim_{t \to \infty} \tilde{\varphi}(sc(t)^{1/\alpha}, t) = \frac{1}{1 + s^{\alpha}} \tag{2.16}$$

The scaling form that is reached for large times is thus determined uniquely by the fractal dimension $\alpha$. For $\alpha = 1/2$, the following explicit result is found:

$$\varphi(x, t) \overset{t \to \infty}{\approx} \left[\frac{c(t)}{\pi x}\right]^{1/2} - c(t)\, e^{c(t)x} \operatorname{Erfc}\{[c(t)x]^{1/2}\} \tag{2.17}$$

## 2.2. Nearest-Neighbor Particle Annihilation

It is well known that diffusion-controlled reactions in low-dimensional systems can be characterized by anomalous reaction kinetics. For example, the reactions $A + A \to A$ or $A + A \to$ products, where the A particles perform Brownian motion on the line and react when they touch, lead to the following long-time behavior of the concentration of A particles:

$$c(t) \sim t^{-1/2} \tag{2.18}$$

This result has to be contrasted with the $c(t) \sim t^{-1}$ that would follow from a classical kinetic rate equation $\partial_t c = -c^2$. The slower decay seen in Eq. (2.18) can be explained by the fact that the reaction produces a self-ordering in the positions of the particles of the following type[10]:

$$\varphi(x, t) = \frac{\pi}{2} c^2(t)\, x e^{-\pi[c(t)x]^2/4} \tag{2.19}$$

In some problems, particles can annihilate via long-range interactions, such as exchange interactions, even in the absence of particle motion. In this case, an exact analytic treatment is difficult. The problem becomes tractable if we assume that particles react only with their nearest neighbors, and that the probability of reaction is independent of the relative distance of the reacting pair. We also assume an initial distribution of particles on the line, with identical independently distributed distances between the successive pairs, with density $\varphi(x, t=0)$. Neighboring particles can react and disappear at a constant rate $R$. An equation of evolution for $\varphi(x, t)$ can now be derived along the same lines as those followed in the previously discussed single-particle annihilation. One finds [compare to Eq. (2.3)]

$$\partial_t \varphi(x, t) = R\left[\int_0^x dx' \int_0^{x-x'} dx''\, \varphi(x', t)\, \varphi(x'', t)\, \varphi(x-x'-x'', t) - \varphi(x, t)\right] \tag{2.20}$$

This equation can again be solved by spatial Laplace transform, and one finds

$$1 - \frac{1}{\tilde{\varphi}^2(s, t)} = e^{2Rt}\left[1 - \frac{1}{\tilde{\varphi}^2(s, t=0)}\right] \tag{2.21}$$

The following conclusions can be drawn:

(1) If the initial density $\varphi(x, t=0)$ has a finite first moment, the distribution converges to the following attractor [compare to (2.11)]:

$$\lim_{t\to\infty} \tilde{\varphi}(sc(t)) = \frac{1}{(1+2s)^{1/2}} \tag{2.22a}$$

with

$$c(t) = e^{-2Rt} c(0) \tag{2.22b}$$

or, in real space,

$$\varphi(x, t) \overset{t\to\infty}{\approx} \left[\frac{c(t)}{2\pi x}\right]^{1/2} e^{-c(t)x/2} \tag{2.23}$$

Note the (normalizable) divergence of $\varphi$ for $x \to 0$. This small $x$ dependence implies that the probability to find a nearest neighbor in the interval $]0, x]$, given that a particle is located at zero, is proportional to $\sqrt{cx}$ for $x \ll c^{-1}$ (if such a behavior were true for all $x$, one would have a fractal with dimension 1/2).

(2) The attractor corresponding to initial densities with fractal dimension $\alpha$ is

$$\lim_{t\to\infty} \tilde{\varphi}(sc(t)^{1/\alpha}, t)) = \frac{1}{(1+s^{\alpha})^{1/2}} \tag{2.24}$$

It is straightforward to generalize the above results to the case of $n$-nearest-neighbor annihilation. The attractor for initial densities with a finite first moment is then

$$\lim_{t\to\infty} \tilde{\varphi}(s/\langle x(t)\rangle) = \frac{1}{(1+ns)^{1/n}} \tag{2.25}$$

with

$$c(t) = e^{-nRt}c(0) \tag{2.26}$$

and in real space

$$\varphi(x, t) = c(t)\,\frac{e^{-c(t)x/n}}{\Gamma(1/n)\, n^{1/n}[c(t)x]^{(n-1)/n}} \tag{2.27}$$

We note that the above type of dynamics leads to strong nearest neighbor correlations, even though next nearest correlations are absent and even though the rate equation is linear in $c$.

## 3. FIRST-PASSAGE-TIME APPROACH TO ONE-DIMENSIONAL BALLISTIC MODELS

A ballistic analogue of the annihilation kinetics $A + A \to$ product was introduced and solved by Elsken and Frisch.[12] In this model, particles move along the line with velocity $+v$ or $-v$, and annihilate upon collision. When both types of particles occur with the same probability $p = q = 1/2$, and the initial distances between neighboring particles form a renewal process with finite average distance, the concentration of A particles goes down asymptotically as $t^{-1/2}$. This can be explained by a central limit type of argument, similar to the one used for explaining the anomalous behavior in the $A + B \to$ product: after a time $t$, particles cover a distance of the order of $t$. However, the number of particles with velocities $+v$ and $-v$ on an interval of length order $t$ will not match exactly, because of fluctuations that are of order $\sqrt{ct}$, where $c$ is the concentration of particles. The particles that happen to be in the majority survive, so that, after time $t$, the number of particles is reduced from order $t$ to order $\sqrt{t}$. In ref. 24 we presented a ballistic analogue of the reaction $\mathrm{A}+\mathrm{A} \rightarrow \mathrm{A}$. The particles can now have a velocity $+v$ (probability $p$), $-v$ (probability $q$), or velocity 0. Upon collision of two moving particles, a stationary particle is produced, while stationary particles are annihilated upon collision with a moving particle. This model gives results similar to the $\mathrm{A}+\mathrm{A} \rightarrow$ product model when $p=q$, but the approach of the stationary state is dominated by the immobile particles for $p \neq q$. We illustrate how both models, and possibly more complicated ballistic models, can be solved in an elegant way by mapping them to a first-passage-time problem for a discrete-time random walk. To illustrate the idea, we concentrate on the simplest case of the annihilation kinetics $\mathrm{A}+\mathrm{A} \rightarrow$ product. More details can be found in ref. 24. Consider a particle that is moving to the right. To find its survival time, one has to locate its collision partner. This can only be a particle at its right-hand side that is moving to the left. It can be identified by the following mapping to a random walk problem. We consider a random walk on the integers, starting at lattice site 1. We consider the subsequent neighbors $n=1, 2,\ldots$ to the right of the particle under consideration. $n$ plays the role of the discrete-time variable of the random walk. When the $n$th neighbor is moving to the right, we take, at time $n$, a random walk step away from the origin. This happens with probability $p$. When the $n$th particle is moving to the left, we are "coming closer" to the collision partner, and we take a step toward the origin. The probability is $q$. The collision partner will correspond to the time step at which the origin is reached for the first time. The elapsed time is then equal to the relative distance between the two particles, divided by the relative velocity $2v$. If the distances between successive particles is characterized by a renewal process with finite average value, we conclude that the survival time is essentially determined by the first passage time for the random walk to go from 1 to 0. For unbiased random walks, it is known that the first passage is a certain event (cf. Section 5). We conclude that in this case, all the particles will ultimately disappear by a reactive collision. However, the first passage times form a fractal time process with fractal dimension 1/2; the probability that a first passage time exceeds $t$ decreases as $t^{-1/2}$. We thus recover the above-cited anomalous decay rate of $t^{-1/2}$, but it is now explained in terms of first-passage-time statistics, and similar in spirit to the arguments explaining the behavior in the diffusion-limited $\mathrm{A}+\mathrm{A} \rightarrow \mathrm{A}$ model.[10] When $p>q$, not all the particles moving with velocity $+v$ decay: a first passage from 1 to 0, with a bias $p>q$ away from the origin, is no longer a certain event; rather, it occurs with a probability equal to $(2p-1)/p$. On the other hand, the particles moving with $-v$ velocity are annihilated exponentially fast, since they are performing a biased random walk biased toward the origin. The fractal nature of the decay process is thus destroyed when $p \neq q$.

## 4. RENORMALIZATION OF CONTINUOUS-TIME RANDOM WALKS, INCLUDING TRAPPING

In this section, we show how a renormalization approach that was formulated for continuous-time random walks on regular and fractal lattices[28-31] can also be used to study the effect of trapping. More precisely, it allows one to calculate the survival probability for a particle performing a random walk on the line in the presence of both perfect and imperfect traps. The imperfect traps are supposed to form a uniform background, characterized by a trapping probability density, that is translational invariant. The perfect traps delimit the region inside which the particle can move without being trapped instantaneously.

We start by deriving the survival time in the absence of perfect traps. The continuous-time random walk is characterized by two probability densities $\psi(\tau)$ and $\varphi(\tau)$. Here $\psi(\tau)$ stands for the probability density that the particle jumps to a new site after spending a time $\tau$ on a given site, while $\varphi(\tau)$ is the probability density that the particle is trapped after a time $\tau$ on a given site. At least one of these events takes place at a certain time $\tau \geqslant 0$; hence

$$\int_0^\infty [\psi(\tau)+\varphi(\tau)]\, d\tau = 1 \tag{4.1}$$

Let $\phi(t)$ denote the probability density that a particle is trapped after a time $t$, irrespective of its location on the lattice. By expressing that the lifetime of a particle can exceed $t$ by doing so after $n$ jumps, $n = 0, 1,...$, we find

$$\begin{aligned}\int_t^\infty \varphi(\tau)\, d\tau = \int_t^\infty [\psi(\tau)+\varphi(\tau)]\, d\tau + \sum_{n=1}^\infty \int_0^t d\tau_1\, \psi(\tau_1) \int_0^{t-\tau_1} d\tau_2\, \psi(\tau_2)\\ \times \cdots \int_0^{t-\tau_1-\cdots-\tau_{n-2}} d\tau_{n-1}\, \psi(\tau_{n-1})\\ \times \int_{t-\tau_1-\cdots-\tau_{n-1}}^\infty d\tau_n\, [\psi(\tau_n)+\varphi(\tau_n)]\end{aligned} \tag{4.2}$$

By Laplace transformation, we conclude that

$$\tilde{\phi}(s) = \frac{\tilde{\phi}(s)}{1-\tilde{\psi}(s)} \tag{4.3}$$

Note that this result does not depend on the geometrical structure of the underlying lattice.

The survival time in the presence of a perfect trap can be calculated using a renormalization procedure. We first consider the simple example of a directed one-dimensional walk (see Fig. 1a), starting at the origin $i=0$. Here $\psi_0(\tau)$ stands for the probability density to jump from a given site $i$ to its right nearest neighbor $i+1$, while $\varphi_0(\tau)$ is the probability density for trapping at site $i$. We now decimate every other site, starting with site 1, and calculate the renormalized quantities $\psi_1(\tau)$ and $\varphi_1(\tau)$. Clearly, the probability density $\psi_1(t)$ to go from site 0 to 2 in a time $t$ is given by

$$\psi_1(t)=\int_0^t \psi_0(\tau)\,\psi_0(t-\tau)\,d\tau \tag{4.4}$$

or

$$\tilde{\psi}_1(s)=\tilde{\psi}_0^2(s) \tag{4.5}$$

On the other hand, $\varphi_1(t)$ is the probability density that trapping occurs at site 0 or 1; hence

$$\varphi_1(t)=\varphi_0(t)+\int_0^t \psi_0(\tau)\,\varphi_0(t-\tau)\,d\tau \tag{4.6}$$

or

$$\tilde{\varphi}_1(s)=\tilde{\varphi}_0(s)[1+\tilde{\psi}_0(s)] \tag{4.7}$$

The same decimation procedure can now be repeated over again, and the renormalization equations, linking the densities $\psi_n$ and $\varphi_n$ after $n$ decima-

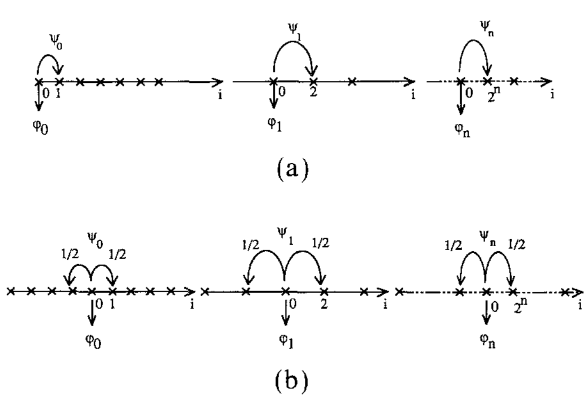


Fig. 1. Decimation procedure for (a) a directed random walk and (b) a nearest neighbor random walk.

tions, to those after $n-1$ decimations, namely $\psi_{n-1}$ and $\varphi_{n-1}$, are given by

$$\tilde{\psi}_n = \tilde{\psi}_{n-1}^2 \tag{4.8a}$$

$$\tilde{\varphi}_n = \tilde{\varphi}_{n-1}[1 + \tilde{\psi}_{n-1}] \tag{4.8b}$$

For convenience, we are not explicitly indicating the $s$ dependence. Note that these equations preserve normalization, since $\tilde{\psi}_{n-1}(s=0) + \tilde{\varphi}_{n-1}(s=0) = 1$ implies that $\tilde{\psi}_n(s=0) + \tilde{\varphi}_n(s=0) = 1$. Also, these renormalization equations can easily be solved by iteration. One finds

$$\tilde{\psi}_n = \tilde{\psi}_0^{2^n} \tag{4.9a}$$

$$\tilde{\psi}_n = \tilde{\varphi}_0 \prod_{k=1}^{n} (1 + \tilde{\psi}_0^{2^{k-1}}) = \tilde{\varphi}_0 \frac{1 - \tilde{\psi}_0^{2^n}}{1 - \tilde{\psi}_0} \tag{4.9b}$$

In particular, it follows from (4.9b) that if $|\tilde{\psi}_0(s=0)| < 1$, i.e., the probability for hopping is not equal to one, then

$$\lim_{n \to \infty} \tilde{\varphi}_n = \frac{\tilde{\varphi}_0}{1 - \tilde{\psi}_0} \tag{4.10}$$

which is in agreement with the general result give in Eq. (4.3). Note also that $P_n = \tilde{\psi}_n(s=0)$ is equal to the probability that a particle has reached the site at a distance $L = 2^n$ from the origin. From Eq. (4.9), we find that this probability decreases exponentially with the distance $P_n \sim \exp(-L/L_0)$ with effective trapping length $L_0 = -1/\ln P_0$.

Let us now turn to the case of a nearest neighbor random walk in one dimension (see Fig. 1b). The probability density to jump from a given site to a specified nearest neighbor at time $\tau$ is now equal to $\psi_0(\tau)/2$. The particle starts at the origin. By decimating all uneven-numbered sites, one obtains the following renormalization equations:

$$\tilde{\psi}_n = \frac{\tilde{\psi}_{n-1}^2}{2 - \tilde{\psi}_{n-1}^2} \tag{4.11a}$$

$$\tilde{\varphi}_n = 2\tilde{\varphi}_{n-1} \frac{1 + \tilde{\psi}_{n-1}}{2 - \tilde{\psi}_{n-1}^2} \tag{4.11b}$$

The solution of recursion relation (4.11a) is well known[29]:

$$1/\tilde{\psi}_n = \cosh[2^n \operatorname{arccosh}(1/\tilde{\psi}_0)] \tag{4.12}$$

while iteration of (4.11b) yields

$$\tilde{\varphi}_n = \tilde{\varphi}_0 \prod_{k=0}^{n-1} 2 \frac{1+\tilde{\psi}_k}{2-\tilde{\psi}_k^2} \tag{4.13}$$

By substituting the result (4.12) for $\tilde{\psi}_k$, we get an explicit, albeit complicated expression for $\tilde{\varphi}_n$ in terms of $\tilde{\varphi}_0$ and $\tilde{\psi}_0$. Furthermore, one can show that

$$\lim_{n\to\infty} \tilde{\varphi}_n = \frac{\tilde{\varphi}_0}{1-\tilde{\psi}_0} \tag{4.14}$$

which is again in agreement with Eq. (4.3).

By setting $s=0$ in Eq. (4.12), we obtain the following result for the probability $P_n$ of reaching one of the sites at $2^n$ or $-2^n$, before trapping $(L=2^n)$:

$$P_n = \frac{1}{\cosh[2^n \operatorname{arccosh}(1/P_0)]} \sim e^{-L/L_0} \tag{4.15}$$

with effective trapping length $L_0 = 1/\operatorname{arccosh}(1/P_0)$.

The above procedure can also be applied in more complicated cases, such as biased random walks or random walks on fractal lattices.[29-31]

## 5. MULTIFRACTALITY OF THE ESCAPE PROBABILITY IN A ONE-DIMENSIONAL RANDOM WALK

When we drop the condition of translational invariance, the previous renormalization approach becomes impractical, at least for an analytic treatment. The trapping probability density at site $i$ now depends on the $i$, $\varphi = \varphi_i$, while one has to distinguish the jump probability densities $\tilde{\psi}_i^+$ and $\tilde{\psi}_i^-$ to go from site $i$ to $i+1$ or $i-1$, respectively. However, it is possible to make some progress using a renormalization approach, similar in spirit to the one presented above, by concentrating on the first-passage-time density $F_i^+(t)$ to go from site $i$ to site $i+1$ for the first time at time $t$.[31-33] A first passage from site $i$ to site $i+1$ can be realized by a number $n$ of excursions to $i-1$, $n=0, 1, 2,\ldots$, each followed by a first passage back to $i$, and finally by a jump from $i$ to $i+1$. In terms of Laplace transforms, one obtains the following result:

$$\tilde{F}_i^+ = \sum_{n=0}^{\infty} (\tilde{\psi}_i^- \tilde{F}_{i-1}^+)^n \tilde{\psi}_i^+ = \frac{\tilde{\psi}_i^+}{1-\tilde{\psi}_i^- \tilde{F}_{i-1}^+} \tag{5.1}$$

In the following, we will concentrate on just one feature of the first passage, namely the probability $P_i = \tilde{F}_i^+(s=0)$ that it ever takes place. We introduce the notation $p_i = \tilde{\psi}_i^+(s=0)$, $q_i = \tilde{\psi}_i(s=0)$, and $r_i = \tilde{\varphi}_i(s=0)$ to designate the probabilities to jump to $i+1$, $i-1$, or to the trapping state from site $i$. By setting $s=0$ in Eq. (5.1), one obtains the following recursion relation:

$$P_i = \frac{p_i}{1 - q_i P_{i-1}} \tag{5.2}$$

For the case of a translational invariant system, $p_i = p$ and $q_i = q$, it is straightforward to find the stable fixed point $P^*$, $0 \leqslant P^* \leqslant 1$, of the recursion relation (5.2), namely

$$P^* = \frac{1 - (1 - 4pq)^{1/2}}{2q} \tag{5.3}$$

The following exact solution of Eq. (5.2) describes the approach to the fixed point $P^*$:

$$P_n = P^* \frac{1 - \alpha((q/p)^{1/2} P^*)^{2n}}{1 - \alpha((q/p)^{1/2} P^*)^{2n+2}} \tag{5.4}$$

where $\alpha$ is determined by the "initial" condition, namely the value of $P_n$ at one of the previous sites, e.g., site $n=0$:

$$P_0 = P^* \frac{1 - \alpha}{1 - \alpha((q/p)^{1/2} P^*)^2} \tag{5.5}$$

Note that an absorbing boundary condition $P_0 = 0$ implies that $\alpha = 1$.

$P_n$ stands for the probability of ever reaching site $n+1$, starting from site $n$. The probability $P_{n,n+i}$ for ever reaching site $n+i$, starting from site $n$, is given by

$$P_{n,n+i} = P_n P_{n+1} P_{n+i-1} \tag{5.6}$$

Hence

$$P_{n,n+i} = (P^*)^i \frac{1 - \alpha((q/p)^{1/2} P^*)^{2n}}{1 - \alpha((q/p)^{1/2} P^*)^{2n+2i}} \tag{5.7}$$

In the absence of trapping $(p+q=1)$, one has that

$$P^* = \frac{1 - (1 - 2p)}{2(1-p)} = \begin{cases} p/(1-p), & p \leqslant 1/2 \\ 1, & p \geqslant 1/2 \end{cases} \tag{5.8}$$

If, in addition, one considers the case of an absorbing boundary condition ($P_0 = 0$ or $\alpha = 1$), Eq. (5.7) reduces to

$$P_{n,n+i} = \frac{1-(q/p)^n}{1-(q/p)^{n+i}} \tag{5.9}$$

a well-known result from the gambler's ruin problem.[27] Note that Eq. (5.7) includes the effect of trapping ($p+q<1$) and of an imperfect absorbing boundary ($P_0 > 0$).

The abrupt transition, induced by a bias, on the probability of first passage [cf. Eq. (5.8)], is well known.[27] In terms of the map (5.2), this transition can be viewed as a transcritical bifurcation at the value $p = 1/2$; see Fig. 2a. For values of $p > 1/2$ the fixed point $P^* = 1$ is stable, for $p < 1/2$ it is unstable, while at the critical point $p = 1/2$ it is marginally stable. This explains the "critical" behavior given in Eq. (5.8). Moreover, Eq. (5.4) takes the following form at the critical point $p = 1/2$:

$$P_n = \frac{P_0 + n(1-P_0)}{1+n(1-P_0)} \tag{5.10}$$

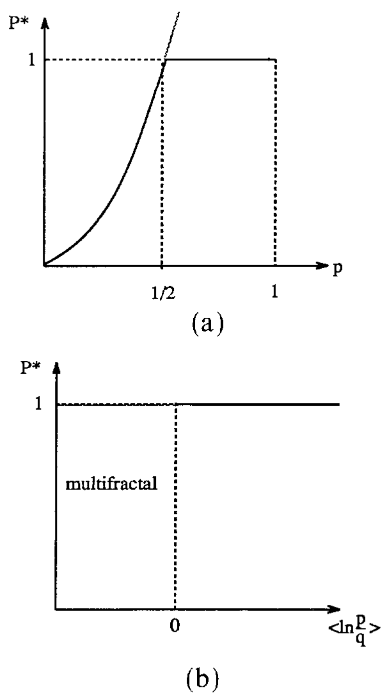



Fig. 2. The bifurcation diagram for the probability for first passage $P^*$, in the absence of trapping ($p+q=1$), for the case (a) without and (b) with disorder.

One observes that the approach of $P_n$ to the fixed-point solution $P^* = 1$ is no longer exponential in $n$, but rather goes as an inverse power law $P_n \approx 1 - 1/n$, for $n$ large. In this sense, the system displays long-range spatial correlations at the "critical point" $p = 1/2$.

By comparing Eqs. (5.1) and (5.2), we note that, in the simple case of a translational-invariant system $\tilde{\psi}_i^+ = \tilde{\psi}_i^- = \tilde{\psi}$, the recursion relation for $P_n$ and $\tilde{F}_i^+$ have an analogous form. The solution for the recursion relation (5.1) is thus given by [compare to Eqs. (5.4) and (5.5)]

$$\tilde{F}_n^+ = \tilde{F}^* \frac{1 - \alpha[(\tilde{\psi}^-/\tilde{\psi}^+)^{1/2}\, \tilde{F}^*]^{2n}}{1 - \alpha[(\tilde{\psi}^-/\tilde{\psi}^+)^{1/2}\, \tilde{F}^*]^{2n+2}} \tag{5.11a}$$

with

$$\tilde{F}^* = \frac{1 - (1 - 4\tilde{\psi}^+\tilde{\psi}^-)^{1/2}}{2\tilde{\psi}^-} \tag{5.11b}$$

where the value of $\alpha$ is again determined by the boundary condition, namely by the function $\tilde{F}_i$ at a boundary site $i$.

We now turn our attention to the case of a random system. For simplicity, we restrict ourselves to the case of uncorrelated binary disorder. At each site $i$, the jump and trapping probability are either $p_1$, $q_1$, and $r_1$ or $p_2$, $q_2$, and $r_2$ (with $p_i + q_i + r_i = 1$ for $i = 1$ or 2), with respective probability $\Pi_1$ and $\Pi_2$, and this independent of the state of the other sites. For example, the system can consist of a random alternation of sites with $p_1 = q_1 = 2/5$, $r_1 = 1/5$ and $p_2 = q_2 = 12/25$, $r_2 = 1/25$ (these are random, but symmetric rates). In Fig. 3, we have plotted the two mappings [cf. Eq. (5.2)], that correspond with each type of site. The question we raise is what will be the values of the escape probability $P_i$ that one observes in the

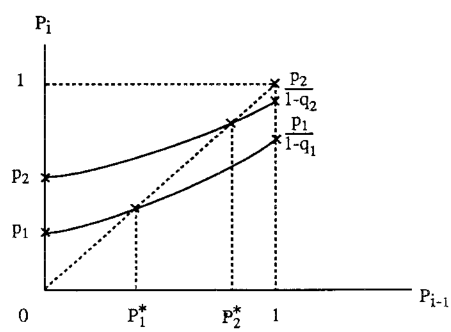
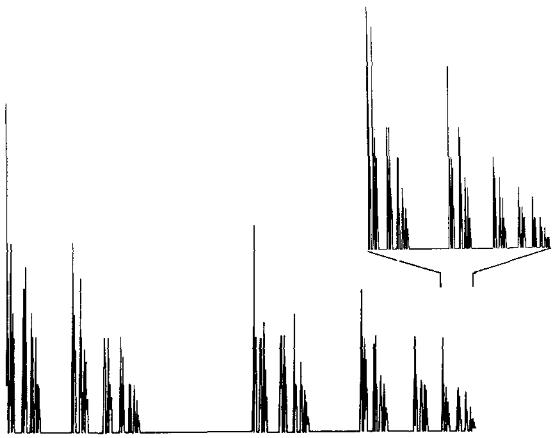
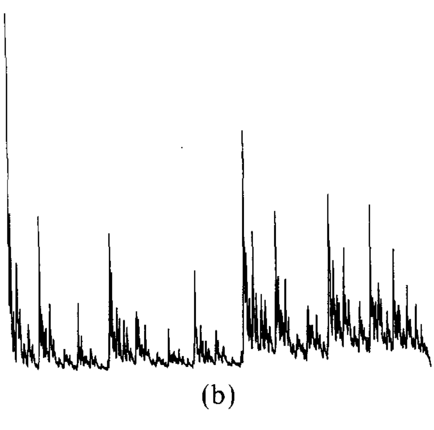



Fig. 3. The two mappings, corresponding to the case of binary disorder $p_1, q_1$ and $p_2, q_2$, given by Eq. (5.2).

limit of an infinite system $i \to \infty$. In a "pure" system consisting of sites of type 1 or 2 only, $P_i$ converges to the corresponding stable fixed points $P_1^*$ and $P_2^*$, as we discussed above. In the random system, however, a new value $P_{i+1}$ will be generated closer to either one of these fixed points, depending on the type of the site encountered. As a result, the escape probability $P_i$ itself becomes a random variable, characterized by an invariant measure or probability density with a support located between $P_1^*$ and $P_2^*$. In Fig. 4a, we have plotted a typical probability profile corresponding to the case of symmetric disorder, namely $p_1 = q_1 = 2/5$ or



(a)



(b)

Fig. 4. The histograms of the multifractal probability density corresponding to the site disorder (a) $p = q = 2/5$ or $p = q = 12/25$ and (b) $p = 3/5$, $q = 2/5$ or $p = 1/5$, $q = 4/5$, with an accuracy of $2^{16}$ cells.

$p_2 = q_2 = 12/25$ (with $\Pi_1 = \Pi_2 = 1/2$). The probability density is of a multifractal nature.[34–38] The generalized dimensions $D_q$ are given in Table I. These results are obtained using the method of transient chaos.[3] For comparison, analytic results are included for the case of the two-scale Cantor set, obtained by replacing the mappings by their tangents in the respective fixed points. In this case, the generalized dimensions $D_q$ are the solution of the following transcendental equation:

$$r_1^{(1-q)\,D_q} + r_2^{(1-q)\,D_q} = 2^q \tag{5.12}$$

where $r_1$ and $r_2$ are the slopes of these linear mappings (also called the contraction rates[35]). The above case is an example of a thin fractal, with a fractal support. It is also possible that the probability profile is a fat fractal, whose support is the whole interval between $P_1^*$ and $P_2^*$ (see below).

In the absence of trapping, $p_i + q_i = 1$, several cases can be distinguished according to the value of $\langle \ln p_i/q_i \rangle$. When $\langle \ln p_i/q_i \rangle > 0$, it is easy to verify that $P_i$ will converge to 1 with probability 1, i.e., first passage is a certain event. For $\langle \ln p_i/q_i \rangle < 0$, one gets a fat multifractal spectrum, of the type represented in Fig. 4b (corresponding to the case $p_1 = 3/5$; $q_1 = 2/5$ or $p_2 = 1/5$; $q_2 = 4/5$). The case of Sinai disorder,[41] defined by the property that $\langle \ln(p_i/q_i) \rangle = 0$ [and $\langle \ln^2(p_i/q_i) \rangle < \infty$], lies at the border-

[3] Data kindly provided by T. Tel.

**Table I. The Generalized Dimensions $D_q$ for the Case $p_1 = q_1 = 2/5$ and $p_2 = q_2 = 12/25$, Obtained from the Solution of the Transcendental Equation (5.12), Corresponding to the Two-Scale Cantor Approximation, and Using the Method of Transient Chaos**[a]

| $q$ | Cantor approximation | Transient chaos |
|---|---|---|
| −10 | 1.09 | 1.09 |
| −8 | 1.07 | 1.07 |
| −5 | 1.00 | 1.00 |
| −2 | 0.87 | 0.84 |
| −1 | 0.81 | 0.77 |
| 0 | 0.75 | 0.71 |
| 1 | 0.71 | 0.67 |
| 2 | 0.67 | 0.64 |
| 5 | 0.60 | 0.59 |
| 8 | 0.56 | 0.56 |
| 10 | 0.55 | 0.55 |

[a] Data kindly provided by T. Tel.

line of these two situations. It was shown by Sinai that the mean square displacement grows very slowly with time, $\langle x^2(t)\rangle \sim \ln^4 t$, in contrast to the normal diffusive behavior that is observed for the case of random but symmetric rates. To illustrate the properties of $P^*$, we consider the case of binary Sinai disorder, i.e., the jump probabilities to the left and the right at a given site are equal to $p$ and $q$ or $q$ and $p$, with equal probability ($\Pi_1 = \Pi_2 = 1/2$). For the mapping with a bias toward the right ($p > q$), the stable fixed point is $P_1^* = 1$. In the vicinity of this fixed point, both mappings can be written in the following form:

$$\ln(1 - P_n) = \ln(1 - P_{n-1}) \pm \ln(p/q) \tag{5.13}$$

In other words, the logarithm of the deviation from the fixed-point value undergoes an unbiased Brownian motion. On the other hand, it is easy to verify that the motion in the vicinity of the stable fixed point $P_2^*$, associated to the mapping $p < q$, is biased away from the latter. We conclude that $P_n$ will converge after a large number of iterations to the final attractor $P_1^* = 1$. The probability density for $P^*$ no longer has a multifractal profile, with a support located between the two fixed points, but rather it is a delta function centered at $P_1^* = 1$: first passage is a certain event in the case of Sinai disorder. In view of Eq. (5.13), the approach to this attractor is not exponentially fast, but is expected to go as

$$1 - P_n \sim \exp\left[-C\left(n \ln^2 \frac{p}{q}\right)^{1/2}\right] \tag{5.14}$$

where $C$ is a constant. The above results remain valid in more complicated cases of Sinai disorder. We conclude that the first passage from a site to one of its neighbors, and hence to any other site, is a certain event in a system with Sinai disorder. The corresponding mean first-passage-time properties have been studied in detail (see, e.g., ref. 31, and references cited therein).

Multifractal properties, such as the ones observed in the above model, are expected to be a general feature of systems in which various forms of disorder are present (see also ref. 42). For example, it is well known that random walks are characterized by fractal properties. On the other hand, disordered lattices can give rise to fractal features, the percolating lattice being a notorious example. The advantage of the above simple model is that the multifractal properties can be studied in relative detail, and their properties can be linked to the well-documented domain of random maps.[35,43,44]

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